Shear-induced rigidity in athermal materials: A unified statistical framework

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Recent studies of athermal systems such as dry grains and dense, non-Brownian suspensions have shown that shear can lead to solidification through the process of shear jamming in grains and discontinuous shear thickening in suspensions. The similarities observed between these two distinct phenomena suggest that the physical processes leading to shear-induced rigidity in athermal materials are universal. We present a nonequilibrium statistical mechanics model, which exhibits the phenomenology of these shear-driven transitions, shear jamming and discontinuous shear thickening, in different regions of the predicted phase diagram. Our analysis identifies the crucial physical processes underlying shear-driven rigidity transitions, and clarifies the distinct roles played by shearing forces and the packing fraction of grains.

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I. INTRODUCTION

Athermal materials such as dry grains and dense non-Brownian suspensions can respond to shear by organizing into structures that support the imposed load [1]: a process that has been termed shear jamming (SJ) in grains [2–5], and discontinuous shear thickening (DST) in suspensions [6–13]. The nature of this self-organization process has been intensely investigated in recent months, and striking similarities have been observed between the two transitions. This is remarkable since the SJ transition occurs through a quasistatic process and refers to static states of particles interacting via purely repulsive contact interactions [2,4,5], and DST occurs through a dynamical process that creates nonequilibrium steady states (NESSs) of particles interacting via hydrodynamic and contact interactions [7,14]. The single most important trigger for these transitions has been identified as the proliferation of frictional contacts [2,7,9]. In a similar range of φ, athermal suspensions exhibit DST as increasing shearing rate leads to a loss of lubrication forces and an increasing number of frictional contacts [7,9].

Lattice models have a venerable history of identifying the core physical mechanisms driving phase transitions, and finding commonalities between seemingly disparate systems. We have constructed a nonequilibrium, driven, disordered model that focuses on the processes of formation and rearrangement of frictional contacts under driving by a field. In contrast to studies that interrogate the microscopic mechanisms leading to the SJ and DST transitions [5,9], we analyze an effective theory that is built on the premise that the driving field, either strain (γ) or strain rate (γ′), increases shear stress and promotes the formation of frictional contacts. We examine the consequences of the interplay between the driving field and the underlying disorder of the contact network on the development of a robust, force-bearing network. The model focuses solely on the force network: changes in the network of frictional contacts with their associated tangential forces strongly affect the viscosity of suspensions in the DST regime [7,9,14]. The mapping between the parameters defining the model and the physical parameters defining and controlling force networks in the SJ and DST transitions are summarized in Table I. In the next section we develop the model starting from a rigorous mapping of grain-level stresses to spins.

II. MODEL

A. Rigorous mapping

The tensor representing the stress state of a grain can be divided into a completely isotropic part that defines hydrostatic pressure and a deviatoric part that represents normal and shear stresses. The deviatoric part can be represented as an element of a vector space [17]. Illustrating in two dimensions (2D), the stress tensor of a grain, which is symmetric since the grain is torque balanced, can be written as

\[ \hat{\sigma} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{pmatrix} \]

(1)

where \( P \) is the hydrostatic pressure, \( \sigma_N = (\sigma_{xx} + \sigma_{yy})/2 \) is the normal stress, and \( \tau = \sigma_{xy} \) is the shear stress. The deviatoric part of the stress, which excludes this hydrostatic part, is therefore an element of a 2D vector space spanned by two \( 2 \times 2 \) matrices, \( \sigma_2 \) and \( \sigma_3 \) [17]. The components of the vector are the normal stress \( \sigma_N \) and the shear stress \( \tau \), and the length \( \sqrt{\sigma_N^2 + \tau^2} \) provides a measure of the stress anisotropy of each grain.

The stress state of a grain is influenced by the local strain arising from the displacement of the neighboring grains. The displacement of the grains is comprised of a homogenous part, which can be characterized by a set of affine transformations and an inhomogeneous part, called nonaffine displacement, which cannot be described through a series of affine transformation of the grain coordinates. The nonaffine displacements are best characterized by a measure called \( D_{\text{an}}^2 \), first introduced by Falk and Langer [18]. To calculate \( D_{\text{an}}^2 \), one measures the actual displacements of the grains, and then chooses an optimum affine strain tensor \( e_{ij} \), which minimizes the mean squared deviation of the actual displacement from
TABLE I. This table demonstrates the mapping between model parameters (Model) and the physical parameters controlled or measured in shear jamming (SJ) experiments and discontinuous shear thickening (DST) simulations.

<table>
<thead>
<tr>
<th></th>
<th>SJ</th>
<th>DST</th>
<th>Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\gamma) (Strain)</td>
<td>Nonaffine strain [15,16]</td>
<td>Nonaffine strain [7,14]</td>
<td>Quenched random field (h_i).</td>
</tr>
<tr>
<td>Microscopic variables: Deviatoric part of stress tensor [see Sec. II A].</td>
<td>Microscopic variables: Deviatoric part of stress tensor [see Sec. II A].</td>
<td>Microscopic variables: (N) Spins which can take three values, +1, 0, or -1.</td>
<td></td>
</tr>
<tr>
<td>(f_{\text{niso}}), the fraction of grains with number of contacts greater than or equal to three. [2,4,15]</td>
<td>(f_{\text{niso}}), the fraction of grains with number of contacts greater than or equal to four. [7,14]</td>
<td>(\langle X \rangle = 1 - \frac{1}{2} \sum S_i^2), the fraction of zero spins.</td>
<td></td>
</tr>
<tr>
<td>Stress anisotropy [4,15]</td>
<td>Stress anisotropy [7,14]</td>
<td>(\langle M \rangle = 1 - \frac{1}{2} \sum S_i), the average magnetization.</td>
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</table>

All the simulations and experiments on DST are in 3D, whereas the SJ experiments are in 2D.

A homogenous displacement due to the strain tensor. This minimized deviation is referred to as \(D_{\text{min}}^2\). The \(D_{\text{min}}^2\) measure has been applied to characterize the nonaffine displacements in granular experiments [16] and shows that the nonaffine strains follow a Gaussian distribution with mean approximately zero. Additionally, it has been found [15] that the deviatoric stress vectors interact with these nonaffine strains in a manner similar to how magnets interact with spins. Thus, the continuous vectors \(\langle \Sigma_{N_i} \rangle\) can be imagined as continuous spins. The vector sum of these spins maps to the global deviatoric stress tensor, and the magnetization measures the stress anisotropy of the global stress tensor.

B. Mapping to a spin-1 Ising model

Although rigorous, analyzing the properties of such a continuous spin model with variable lengths in the presence of a random field is difficult. We therefore use a threshold to map the grain-level stress to a spin-1 Ising model. Let us define \(\Sigma_{\text{dev}} = \sqrt{\Sigma_{N_i}^2 + r^2}\). If \(\Sigma_{\text{dev}}/P \ll 1\), we map the grain to \(S = 0\); otherwise we map it to \(S = \pm 1\), depending on whether the grain points along or perpendicular to the compressive strain direction. The mapping is illustrated in Fig. 1 for the two-dimensional SJ system. In such a system, \(S_i = \pm 1\) represent grains with two contacts, which have strong stress anisotropy \((\Sigma_{\text{dev}}/P = 1)\), and \(S_i = 0\) represent grains with more than two contacts, which have a nearly isotropic stress tensor \((\Sigma_{\text{dev}}/P \ll 1)\) and connect chainlike force networks. A similar mapping applies to the three-dimensional DST systems, with \(S_i = 0\) referring to grains connecting chainlike networks [7].

We envision the SJ and DST processes as ones where the contact network and grain-level stresses reach a force- and torque-balanced state in the presence of driving [5,7,14]. We model this by zero-temperature, single-spin-flip, energy-minimizing dynamics of the energy function (Fig. 1) [19–21]:

\[
\mathcal{H} = -J \sum_{\langle i,j \rangle} S_i S_j - \sum_i h_i S_i - H \sum_i S_i + \Delta(H \sum_i S_i^2). \tag{4}
\]

In SJ experiments, it is known that as the imposed strain \(\gamma\) is increased, the fraction of grains with small stress anisotropy \((S = 0\) in our model) increases. In DST, it is the strain rate that plays the same role. Our viewpoint is that this is the primary effect of \(\gamma\) or \(\dot{\gamma}\). So we map \(H\) to \(\gamma\) for SJ and to \(\dot{\gamma}\) for DST, whereas spin flips map to rearrangements of the contact network of particles. The \(H\)-dependent chemical potential \(\Delta(H)\) incorporates the effect of \(\gamma\) or \(\dot{\gamma}\) on the fraction of grains with small stress anisotropy in SJ (DST). In these systems with shear-induced rigidity, this fraction increases with increasing driving, and we therefore restrict our analysis to increasing functions of \(H\). However, the model is more general and can address other scenarios.

As seen in Fig. 1, shearing leads to significant nonaffine displacements: displacements that are inhomogeneous, and cannot be described by any type of homogeneous deformation of the unstrained state [16]. These are represented by the random magnetic field \(h_i\) at every site. SJ experiments indicate that the distribution of the \(D_{\text{min}}^2\) depends on \(\phi\) but evolves little during the shear-jamming process [5,15]; therefore, we treat \(h_i\) as a quenched random field chosen from a Gaussian distribution with zero mean and variance \(\gamma^2\), as well as an external field \(H\).

![FIG. 1. (Color online) Mapping to spin model: (a) A typical sheared packing undergoing the SJ transition [15], color coded according to the strength of the nonaffine strain \((D_{\text{min}}^2)\) at a grain. (b) Mapping to spin-1 Ising variables: grains with more than two contacts (green) are assumed to have \(S = 0\) (stress anisotropy below threshold), and grains with two contacts have either \(S = 1\) (red) or \(S = -1\) (blue), depending on whether the contact is aligned along the compressive [yellow broken line in (a)] or dilational direction. (c) Enlargement of a small section of (a) illustrating the grain-spin mapping. (d) A schematic configuration of the spin model on a square lattice, color coded by the strength of \(h_i\), which represents the nonaffine strain at site \(i\). The external field \(H\) is not shown.](image)
distribution with zero mean and standard deviation \( R \). In granular systems, the constraints of mechanical equilibrium introduce effective interactions between the stress tensors of grains. In a force chain where every grain has only two contacts, the anisotropies of the stresses of grains in the chain are highly correlated [5]. We model this effective interaction by a ferromagnetic interaction between spins.

We are interested in understanding the effects of shear in creating robust force networks through the introduction of frictional contacts. Our model, therefore, differs from other driven-disordered models in the class of Eq. (4) [21] in one crucial respect: the external field controls the average of the force network is through the driving field. In this work, we focus on the aspects of the model that are relevant to shear-induced rigidity; however, the statistics of avalanches and the yielding behavior exhibit interesting additional features, which will be studied in the future.

For \( N \) spins, we define two global order parameters:

\[
\langle X \rangle = 1 - \frac{1}{N} \sum_i \langle S_i^2 \rangle \quad \text{and} \quad \langle M \rangle = \frac{1}{N} \sum_i \langle S_i \rangle.
\]  

(5)

Here, \( \langle X \rangle \) corresponds to the fraction of grains with isotropic stress tensors \( \langle f_{\text{iso}} \rangle \), and \( \langle M \rangle \) corresponds to the stress anisotropy (contact-stress anisotropy) in SJ (DST). The zero-temperature dynamics samples the metastable states of this disordered model, which we associate with the force networks sampled in the SJ and DST processes. In the SJ context, the \( H \) history represents a \( \gamma \) history, while in the DST context, it represents a \( \gamma \) history. Since there are no thermal fluctuations in our model, averages \( \langle \cdot \rangle \) are over metastable states corresponding to different realizations of the quenched disorder field \( \{ h_i \} \). To simplify notation, we eliminate the \( \langle \cdot \rangle \) symbol in the following.

Starting from a metastable state, if \( H \) is changed adiabatically such that all spins that can lower their energy by flipping do so, there could be a range of \( H \) over which the original state is stable. At a certain \( H \), however, a threshold is crossed at some site \( i \) (determined by the \( h_i \) and the effective field \( \sum_j J_{ij} S_j \)) and that spin changes its state. This, in turn, could lead to the threshold being crossed at other sites, creating a cascade of spin flips in an avalanche [21] until a new metastable state is reached.

In the granular context, we envision this exploration of metastable states as corresponding to exploration of force networks that are in local mechanical equilibrium under driving. The SJ process is quasistatic and since the nonaffine strain field is observed to evolve only weakly over the range of \( \gamma \) probed by the experiments, there is a clear correspondence between the sampling of metastable states in the model and the force networks in the granular assembly. In DST, however, one studies time averages in the NESS at a given \( \gamma \). The correspondence between the ensemble average over \( \{ h_i \} \) and the time average is valid if the NESS dynamically samples nonaffine strains with Gaussian statistics, and if the time taken to reach a force- and torque-balanced state is much shorter than the relaxation time of the nonaffine strain field. These assumptions are validated in simulations [22]. The adiabatic assumption implies that \( \gamma \) is ramped up slowly compared to microscopic time scales [6].

A priori, it is not clear what experimental knob can be turned to tune \( R \). However, there are strong arguments presented below, based on comparing predictions of our model to existing experimental and numerical observations, linking increasing \( \phi \) to a reduction in \( R \). A scaling description of DST has been constructed by invoking a stress-scale dependence of the packing fraction at which the viscosity diverges [9]. We relate the stress scale to \( X(R, H) \), and therefore in our approach it is the stress scale that is controlled by \( \phi \), through \( R \), and by \( \gamma \). If the dominant effect of \( \phi \) on the force network is through the statistics of the nonaffine strain field, then the two approaches should yield similar results. Below, we will establish specific \( \phi \rightarrow R \) mappings in the SJ and DST regimes by comparing our predictions to experiments, simulations, and the scaling theory.

### III. RESULTS

#### A. Mean-field solution of the model

To solve the spin-1 Ising model under the mean-field (MF) approximation, we observe that the order parameters can be represented through the probability of finding a particular value of spin at a particular lattice point:

\[
1 - X = \frac{1}{N} \sum_i \langle S_i^2 \rangle = P(S_i = 1) + P(S_i = -1),
\]

(6)

\[
M = \frac{1}{N} \sum_i \langle S_i \rangle = P(S_i = 1) - P(S_i = -1),
\]

(7)

where \( P(S_i = x) \) measures the probability that the \( i \)th spin takes the value \( x \) (±1 or 0). Also, in the MF approximation, the energy of a spin \( S_i \) is given by

\[
E(S_i) = -JMS_i - HS_i - h_i S_i + \Delta S_i^2.
\]

(8)

Therefore,

\[
E(S_i = 1) \equiv E_1 = -(JM + H + h_i) + \Delta,
\]

\[
E(S_i = -1) \equiv E_{-1} = (JM + H + h_i) + \Delta,
\]

\[
E(S_i = 0) = 0.
\]

In our zero-temperature dynamics, a spin \( S_i \) will be in the +1 state if \( E_1 < 0 \), and \( E_{-1} < -E_1 \). This condition is satisfied if

\[
h_i > \begin{cases} 
\Delta - JM - H & \text{if } \Delta > 0, \\
-JM - H & \text{if } \Delta \leq 0,
\end{cases}
\]

whence

\[
P(S_i = 1) \equiv P(1) = \begin{cases} 
\frac{1}{2} \text{erfc} \left( \frac{\Delta - JM - H}{\sqrt{4R}} \right) & \text{if } \Delta > 0, \\
\frac{1}{2} \text{erfc} \left( \frac{H - JM}{\sqrt{4R}} \right) & \text{if } \Delta \leq 0.
\end{cases}
\]

(9)

A similar calculation yields

\[
P(S_i = -1) \equiv P(-1) = \begin{cases} 
\frac{1}{2} \text{erfc} \left( \frac{\Delta + JM + H}{\sqrt{4R}} \right) & \text{if } \Delta > 0, \\
\frac{1}{2} \text{erfc} \left( \frac{-JM + H}{\sqrt{4R}} \right) & \text{if } \Delta \leq 0.
\end{cases}
\]

(10)
Using these probabilities, and the definitions of $M$ and $X$ [Eq. (6)], we obtain

$$M = \begin{cases} \frac{1}{2} \text{erf} \left( \frac{\Delta H + H - M}{\sqrt{2} \sigma} \right), & \Delta > 0, \\ \text{erf} \left( \frac{H + M}{\sqrt{2} \sigma} \right), & \Delta \leq 0, \end{cases}$$

$$X = \begin{cases} \frac{1}{2} \text{erf} \left( \frac{\Delta H + H + M}{\sqrt{2} \sigma} \right) + \text{erf} \left( \frac{\Delta H - H - M}{\sqrt{2} \sigma} \right), & \Delta > 0, \\ 0, & \Delta \leq 0. \end{cases}$$

Here erf and erfc are the error function and the complementary error function, respectively. Special cases and several important aspects of the MF solution are detailed in the Appendix.

**B. Mean-field phase diagram**

Mean-field calculations of $X$ and $M$ along a forward trajectory, with monotonically increasing $H$ [Fig. 2(a)], suffice to illustrate that the phenomenology of both the SJ and DST transitions are realized in the model. The mean-field phase

![Diagram](image)

FIG. 2. (Color online) Model behavior: (a) A typical forward shear trajectory from the model for $\alpha = 4$ demonstrating the appearance of $M_{\text{peak}}$ in the $M(H)$ (red, continuous line) plot, which is concomitant with saturation of $X(H)$ (blue, dashed line). (b) Mean-field phase diagram for $\alpha = 4$: The color bar indicates $M_{\text{peak}}$. The critical point $(\Delta_x, R_x)$ (yellow circle) marks the end point of three transition lines (see text): $R_x(\Delta_0)$ (black, continuous line), $R_{\text{DST}}(\Delta_0)$ (light blue, dotted line), and $R_{\text{iso}}(\Delta_0)$ (white dashed line). Detailed descriptions of $R_x$, $R_{\text{DST}}$, and $R_{\text{iso}}$ are presented in the Appendix. The region to the left of the pink dash-dotted line has $M_{\text{peak}} \leq 10^{-10}$ and the region to the right of $R_{\text{iso}}$ has $M_{\text{peak}} \sim 1$. (c),(d) $M(X)$ at different values of $R$ and $\Delta_0$ can be used to characterize and distinguish between different shear-induced rigidity transitions. The different colors correspond to the values of $R$ indicated in the phase diagram. For $\Delta_0 > \Delta_x$, (c) $M$ vs $X$ is nonmonotonic, whereas for $\Delta_0 < \Delta_x$, (d), the functional form changes from nonmonotonic to monotonic as $R$ is decreased. $R$ values beyond $R_x$ are marked by dashed lines.

![Diagram](image)

FIG. 3. (Color online) Scaling of $M_{\text{peak}}$: The system achieves peak anisotropy $M_{\text{peak}}$ at the rigidity transition. In (a) we show the dependence of $M_{\text{peak}}$ on $R$ for several $\Delta_0$ [legend in (b)]. For $\Delta_0 > \Delta_x = \sqrt{2/\pi} \sigma \approx 0.4839$, the peak value $M_{\text{peak}}$ is continuous but for $\Delta_0 < \Delta_x$, $M_{\text{peak}}$ has a discontinuity at $R_x$. (b) $M_{\text{peak}}$ has a scaling form as a function of $R/R_{\text{J}}$ with $R_x(\Delta_0) = \Delta_0/6$. For $\Delta_0 < \Delta_x$, the scaling form is valid for $R \leq R_x$, and the discontinuity at $R_x$ is evident in this scaling plot. The peak anisotropy at $R_x$ is $\approx 0$, which suggests that the system undergoes a rigidity transition without going through any anisotropic state, reminiscent of the approach to the isotropically jammed state [2].

![Diagram](image)

FIG. 4. (Color online) SJ regime ($\Delta_0 = 0.9$): (a) $X(H)$ and (b) $M(x)$ for $R = 0.3$ (blue, continuous line), 0.4 (green, long-dashed line), 0.57 (orange, dash-dotted line), 0.84 (red, short-dashed line), and 1.31 (brown, dotted line). The disorder values are chosen in such a way that $R/R_{\text{J}} - 1$ increases logarithmically between 1 and 10. (c),(d) Plots (see text) of $g_x(H/H_{\text{peak}}(R))$ (main figure) and $g_y(H/H_{\text{peak}}(R))$ (inset): $H_{\text{peak}} \sim (R/R_{\text{J}} - 1)^{1/2}$. (d) The scaling functions for $f_{\text{iso}}$ (main) and stress anisotropy (inset) obtained from the SJ experiments [15] with packing fractions much below $\phi_{\text{J}}[0.02 \leq (1 - \phi/\phi_{\text{J}}) \leq 0.09]$. The experimental data show exactly the same scaling behavior as the model [cf. (c)] if $R/R_{\text{J}} - 1$ is mapped to $1 - \phi/\phi_{\text{J}}$ and $H$ is mapped to $\gamma$. 
diagram in the \( R - \Delta \) space is shown in Fig. 2(b). Increasing \( \Delta_0 \) corresponds to higher average concentration of zero spins at \( H = 0 \), corresponding to larger values of \( f_{iso} \) at zero driving field. In the SJ system \([5]\), \( f_{iso}(\gamma = 0) \approx 0.2 \), which maps onto the upper part of the phase diagram in Fig. 2(b). In DST, however, there are a vanishing number of frictional contacts at \( \gamma = 0 \) \([7,14]\), which maps these systems to the lower part of the phase diagram. The \( R \) vs. \( \phi \) mapping discussed earlier implies that \( \phi \) decreases from left to right in Fig. 2(b).

Another parameter that influences the model phase diagram is \( \alpha \), the rate of increase of \( \Delta \) with \( H \). As shown in Fig. 8 in the Appendix, only the \( \alpha > 1 \) protocols lead to a monotonous increase of \( X(H,R) \) for all \( R \), a feature of the number of frictional contacts in both SJ and DST. We, therefore restrict our analysis to \( \alpha > 1 \), and unless otherwise stated the results presented are for \( \alpha = 4 \). For quantitative comparisons to experiments and simulations, one should obtain \( \alpha \) by comparing the mean-field predictions for \( X(H,R) \) to the increase of \( f_{iso}(\gamma)/\langle \gamma \rangle \) in SJ (DST) systems at different \( \phi \).

The qualitative differences between different regions of the phase diagram are best characterized by \( M(X) \), which maps onto the dependence of stress anisotropy on \( f_{iso} \). As shown in Fig. 2(c), for \( \Delta_0 \gg \Delta_\epsilon \), \( M(X) \) has a peak (\( M_{peak} \)) at \( X_{peak}(\Delta_0,R) \), which approaches 1 as \( R \) is decreased, while at the same time \( M_{peak}(\Delta_0,0) \rightarrow 0 \). This prediction of the model is borne out by experimental SJ results, which show the same behavior with increasing \( \phi \). The weak dependence of \( M(X,R) \) on \( R \) for \( X > X_{peak} \) is consistent with experiments \([15]\), where this regime has very weak dependence on \( \phi \). In the limit of small \( \Delta_0 \), the DST regime of the model, the functional form of \( M(X) \) changes with \( R \), as shown in Fig. 2(d). As we discuss below, DST occurs in the \( \phi \) range corresponding to \( R \gtrsim R_{DST} \), where \( M \) is a monotonically decreasing function of \( X \), which explains the monotonic decrease of the stress anisotropy with \( f_{iso} \) observed in numerical simulations \([7,14]\).

### C. Scaling and hysteresis in the SJ regime

The MF calculation shows that at small \( R \) and \( \Delta_0 \gg \Delta_\epsilon \), \( M(H,R) = 0 \) and \( X(H,R) = 1 \) for any \( H \). Physically, this region corresponds to a system in which there are a large number of contacts even at zero driving. The peak anisotropy vanishes as \( M_{peak}(\Delta_0,R) \propto g_{peak}(R/R_J(\Delta_0) - 1) \) (Fig. 3), identifying \( R_J \) as the only characteristic disorder scale in this regime. The two order parameters \( M \) and \( X \) are functions of both \( H \) and \( R \). However, as shown in Fig. 4(c), upon definition of an \( R \)-dependent characteristic field \( H_{peak}(R) \propto R/R_J \) at \( R = R_J \), they obey a scaling form \( X_\epsilon(R,H) = g_\epsilon(H/H_{peak}(R)) \) and \( M_\epsilon(R,H) = g_M(H/H_{peak}(R)) \), where \( X_\epsilon \) and \( M_\epsilon \) are scaled variables: \( x_\epsilon \equiv x - x_{min}^{\epsilon} \). The implication of this is that in the \( \Delta_0 > \Delta_\epsilon \), the behavior at different disorder strengths \( R \) is controlled by the physics of the point \( H = 0 \), \( R_J \), which is reminiscent of critical phenomena \([23]\). It was hypothesized by Bi et al. \([2]\) that \( (\gamma = 0, \phi = \phi_J) \) is a critical point marking the end of a line separating fragile and SJ states. The critical point was characterized by the vanishing of an order parameter which measures the anisotropy of the stress tensor. The current results, based on the spin model, are consistent with that picture. Numerically, the mean-field approximation predicts \( \delta = 1.2 \), and this exponent collapses the experimental data for stress anisotropy and \( f_{iso} \) during a forward shear run \([15]\), if we identify \( R_J \) with \( \phi_J \) [Fig. 4(d)].

The SJ experiments exhibit the phenomenon of Reynolds pressure \([4]\): pressure increasing quadratically with shear strain at small strains, with a Reynolds coefficient that depends only on \( \phi \) and appears to diverge at \( \phi_J \) (Table II). Very general arguments lead to the quadratic dependence of the pressure on shear strain \([24]\). If we make the logical assumption that the pressure increase is determined completely by \( f_{iso} \), and that pressure increases as some monotonic function of \( f_{iso} \) and hence \( X \), then our model provides a natural explanation for the observed \( \phi \) dependence of pressure. The scaling form of \( X(H,R) \) implies that the pressure scales as...
noted that, in the simulation, the values of the model in 2D exhibit hysteresis in this regime. It is to be line), and 0.1 ∼ dotted line), 0.3 (green, dashed line), 0.2 (light blue, dash-dotted line), H is varied cyclically between −0.5 and 0.5. (b) The area of the hysteresis loops exhibits a nonmonotonic behavior and decreases with increasing disorder value beyond a peak. (c) For Δ0 < Δc, the mean-field solution exhibits hysteresis for Rm ≤ R ≤ RDST. A few representative hysteresis curves are shown for R = 0.4 ∼ RDST (red, dotted line), 0.3 (green, dashed line), 0.2 (light blue, dash-dotted line), and 0.1 ∼ Rm (blue, continuous line). The hysteresis loop first appears at RDST, and increases in size as Rm is approached from above. Below Rm, no hysteresis loops exist. (d) The size of the hysteresis loop [measured as H − H0, where H0 (H) is the maximum (minimum) value of H, where a loop exists] increases as R is decreased from RDST. The size increases as a power law with exponent 3/2 [9].

\[ P(R,H) \sim f(X(R,H)) \propto g_P(H/H_{peak}(R)), \]

where \( g_P(x) \) is a scaling function similar to \( g_X \) defined above. The crucial feature of the scaling argument is the vanishing of \( H_{peak}(R) \) as \( R \to R_f \). From symmetry arguments, the pressure has to increase as some even function of the shear strain \( \gamma \) [24] (in the model), \( g_P(x) \) increases at least as fast as quadratically with \( x \) for \( x \ll 1 \). Combined with the scaling form, this argument implies a divergence of the Reynolds coefficient as some power of \( 1/H_{peak} \), and therefore as proportional to \( (R/R_f - 1)^{-\delta_P} \), where \( \delta_P \) depends on the exponent \( \delta \) and the form of \( g_P(x) \) for small \( x \). From the perspective of the model, the source of the divergence observed in experiments is, therefore, directly related to the rapid rise in the number of contacts with shear strain as \( \phi \) increases towards \( \phi_f \); a feature that is consistent with experimental observations.

In the mean-field approximation, there is no hysteresis for \( \Delta_0 \geq \Delta_c \). As we show in Fig. 5(a), numerical simulations of the model in 2D exhibit hysteresis in this regime. It is to be noted that, in the simulation, the values of \( \Delta_0 \) and \( R \) which define the SJ regime differ from the MF calculations. However, the overall structure of the phase diagram remains unchanged, as shown in Fig. 6. The model predictions for the scaling of the hysteresis loops [Fig. 5(b)] with \( R \) are summarized in Table II, and compared to the \( \phi \) dependence observed in SJ experiments [4].

It is clear from the phase diagram that the behavior of the model is completely smooth in the regime \( \Delta_0 \geq \Delta_c \): all properties are continuous but sharp changes occur in the order parameters. This suggests that the SJ transition in dry grains with frictional coefficient \( \approx 1 \) [15] is not a phase transition but a crossover phenomenon at which the contact force network changes continuously as a function of both \( \phi \) and \( \gamma \). Preliminary analysis of experiments with a lower friction coefficient between grains [25] suggests that with decreasing friction coefficient \( \Delta_0 \) approaches \( \Delta_c \) from above, which leaves open the possibility of a true transition.

D. Scaling and hysteresis in the DST regime

In the low-\( \Delta_0 \) regime, mean-field analysis predicts multiple solutions to \( M(H,R) \) and hysteresis under cyclic driving. We can identify three lines based on the multiplicity of solutions: For \( R_m(\Delta_0) \leq R \leq R_c(\Delta_0) \), (i) the mean-field analysis predicts two solutions for \( M(H = 0,R) \) with accompanying hysteresis; (ii) for \( R_m < R < R_{DST}(\Delta_0) \), multiple solutions appear for \( X(H,R) \) leading to multiple hysteresis loops, as shown in Fig. 7. As seen in Fig. 2(b), there is a critical point, \( (\Delta_c, R_c) \), marking the end of these three transition lines. The \( R_{DST} \) and the \( R_m \) lines are present in numerical simulations in 2D but the \( R_c \) line is a mean-field feature. Simulations exhibit hysteresis over most of the region in Fig. 2(b); however, their characteristics change at \( R_{DST} \) and \( R_m \). The \( R_m(\Delta_0) \) line marks a discontinuous transition at which the peak anisotropy decreases dramatically, as shown in Figs. 3 and 7.

Identifying \( R_m \) with the largest packing fraction \( \phi_m \) at which one can have any flow [9], and \( R_{DST} \) with the smallest packing fraction \( \phi_{DST} \) for the onset of DST, our results imply that two distinct types of force networks are stable in suspensions with \( \phi_m > \phi > \phi_{DST} \): one with small stress anisotropy and large \( f_{iso} \) creating a highly connected network of force-bearing linear structures, and one with larger stress anisotropy and
indicate that the avalanche distribution exhibits a power
below \( \Delta_1 \) and \( \Delta_0 \) are the special disorders for \( \Delta = 0.2 \) (see section 3 in Appendix). At \( R_m \), the peak value of \( M \), \( M_{\text{peak}} \), changes discontinuously [Fig. 3(a)]. (a) \( R = 0.09 \) (just below \( R_m \)), (b) \( R = 0.1 \) (just above \( R_m \)), (c) \( R = 0.3 \) (below \( R_{\text{DST}} \)), (d) \( R = 0.4 \) (\( R_{\text{DST}} \)), (e) \( R = 0.6 \) (\( R_{\text{DST}} < R < R_c \)), and (f) \( R = 0.9 \) (\( R > R_c \)).

Smaller \( f_{\text{iso}} \). The networks with large \( f_{\text{iso}} \) also have large pressures since in our picture, the pressure is determined by \( X \).

Mean-field analysis shows that the \( X(H,R) \) hysteresis loops span a range \( \{H_-,H_+\} \), which grows as \( |R - R_{\text{DST}}|^{3/2} \) for \( R \leq R_{\text{DST}} \), and \( H_+ \to 0 \) at \( R_m \). These observations are in accord with scaling predictions of hysteresis loops in DST [9] if we associate \( R_{\text{DST}} \) with \( \phi_{\text{DST}} \).

FIG. 7. (Color online) Hysteresis of \( M \) and \( X \) illustrating the behavior of the model for different disorders (\( \alpha = 4 \) and \( \Delta_0 = 0.2 \), \( R_m \sim 0.097 \), \( R_{\text{DST}} \sim 0.4 \), and \( R_c \sim 0.8 \) are the special disorders for \( \Delta = 0.2 \) (see section 3 in Appendix). At \( R_m \), the peak value of \( M \), \( M_{\text{peak}} \), changes discontinuously [Fig. 3(a)]. (a) \( R = 0.09 \) (just below \( R_m \)), (b) \( R = 0.1 \) (just above \( R_m \)), (c) \( R = 0.3 \) (below \( R_{\text{DST}} \)), (d) \( R = 0.4 \) (\( R_{\text{DST}} \)), (e) \( R = 0.6 \) (\( R_{\text{DST}} < R < R_c \)), and (f) \( R = 0.9 \) (\( R > R_c \)).

IV. DISCUSSION

We have constructed a driven, disordered, zero-temperature (nonequilibrium) statistical mechanics model, which captures all essential features of shear-induced rigidity transitions in granular materials and dense athermal suspensions. Our analysis highlights the distinct roles played by density and frictional contacts. We are beginning to explore our model in this regime, where \( X \approx 1 \) and independent of \( H \).

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APPENDIX

In the following sections we discuss several properties and important aspects of the MF solution. In particular, we discuss the special disorders which define different boundaries of the MF phase diagram. The special case of \( \alpha = 1 \) trajectories is also discussed here.

1. Zero-disorder behavior

The energy of a spin \( S_i \) in the zero-disorder limit is

\[
E(S_i) = -J M S_i - H S_i + \Delta S_i^2. \tag{A1}
\]

If \( S_i = 1 \), and if \( \Delta \gg J M + H \), it will flip to the \( S_i = 0 \) state, and vice versa. A similar calculation can be done for \( S_i = -1 \). Hence, at zero disorder there is a discontinuous transition from \( M = 0 \) to \( M = 1 \).

2. Asymptotic behavior of the model

The asymptotic, large-field behavior of the model is governed by the last two terms in Eq. (4). Thus, the effective model governing the behavior at large positive \( H \) with \( \Delta(H) = \alpha |H| + \Delta_0 \) can be written as

\[
\lim_{H \rightarrow +\infty} \mathcal{H} = -H \sum_i S_i + \Delta(H) \sum_i S_i^2 \tag{A2}
\]

\[
= -H \sum_i S_i + (\alpha |H| + \Delta_0) \sum_i S_i^2. \tag{A3}
\]

The first term in Eq. (A2) favors production of \( S = +1 \) when \( H \rightarrow +\infty \), whereas the second term favors production of \( S = 0 \) when \( H \rightarrow +\infty \). Since \( \Delta \) depends on \( H \), the asymptotic behavior of the model crucially depends on the functional dependence of \( \Delta \) on \( H \), which we refer to as a protocol. For a linear protocol as in Eq. (A3), which is the only kind we have analyzed, the asymptotic behavior depends on the slope \( \alpha \). If \( \alpha > 1 \), \( \Delta \) dominates \( H \), and \( S_i = 0 \) \( \forall \ i \). Conversely, if \( \alpha < 1 \), \( H \) dominates \( \Delta \), and \( S_i = +1 \) \( \forall \ i \). If \( \alpha = 1 \), there is

\[ S_i = 0 \text{, even if the field is being increased monotonically, and the energy at a site does not approach the “flip” threshold monotonically. Recent studies [27] show that this feature affects the yielding transition, suggesting that our model is relevant for understanding the yielding of athermal materials. We have focused on the shear-induced rigidity aspect of athermal, particulate systems. Yielding of the jammed states presumably occurs when the number of frictional contacts is saturated, and shearing does not lead to formation of new contacts. We are beginning to explore our model in this regime, where } X \approx 1 \text{ and independent of } H.\]
The saturation value depends on the disorder.

no $H$ dependence and the asymptotic behavior depends on other terms in Eq. (4). We discuss the $\alpha \leq 1$ trajectories in section 4 of Appendix.

3. Special disorders for $\alpha > 1$ trajectories

The mean-field equations for $\alpha > 1$ and $\Delta_0 < \Delta_c$ admit three lines of transitions which end at a critical point $(R_c, \Delta_c)$. These lines are defined by $R_i(\Delta_0)$, $R_{DST}(\Delta_0)$, and $R_m(\Delta_0)$ in descending order of magnitude [Fig. 1(a) in the main text]. The line $R_i(\Delta_0)$ marks the transition from a single solution for $M(H)$ for $R > R_i(\Delta_0)$ to multiple solutions over a range of $H$ (Fig. 7), whereas the line $R_e(\Delta_0)$ marks the transition from multiple solutions for $M(H)$ with $M_{\text{peak}} \approx 1$ for $R = R_{DST}(\Delta_0)$ to a single solution with $M_{\text{peak}} \approx 0$ for $R = R_m(\Delta_0)$, as shown in Fig. 7. Notably, $M_{\text{peak}}$ has a discontinuity at $R_m(\Delta_0)$ with the discontinuity increasing as $\Delta_0 \to 0$, as seen in Fig. 3. The transition at $R_i(\Delta_0)$ is continuous. The transition lines $R_i(\Delta_0)$ and $R_m(\Delta_0)$ can be calculated analytically from the mean-field equations, yielding

$$R_i(\Delta_0) = \sqrt{-\frac{-\Delta_0^2}{W(0, -\frac{\pi \Delta_0}{2})}}$$

and

$$R_m(\Delta_0) = \sqrt{-\frac{-\Delta_0^2}{W(-1, -\frac{\pi \Delta_0}{2})}}.$$  

Here $W(k, x)$ is the product logarithmic function, also known as Lambert’s $W$ function.

The transition at $R_{DST}(\Delta_0)$ is a unique feature of our model and marks the onset of multiple solutions to $X(H)$, accompanied by system-size avalanches in which spins flip from ±1 to 0. $R_{DST}(\Delta_0)$ is difficult to calculate analytically, and the line shown in Fig. 1(a) of the main text has been obtained numerically. Apart from $\Delta_0$ very close to $\Delta_c$, $R_{DST}(\Delta_0) \approx 0.4$. Figure 7 illustrates the behavior of the system near these special disorders by comparing the $M$ hysteresis. In the main text, these special disorders have been related to special packing fractions relevant to the DST transition.

4. $\alpha \leq 1, \Delta \geq \Delta_c$ trajectories

For $\alpha < 1$, $M$ increases and $X$ decreases as $H$ is increased, indicating that $S_i = \pm 1$ proliferate [Fig. 8(a)]. This trajectory is, therefore, not relevant to shear-induced rigidity where grains with three or more contacts ($S_i = 0$) proliferate, as the system is driven towards jamming.

Lying between $\alpha < 1$ and $\alpha > 1$ trajectories, $\alpha = 1$ trajectories exhibit an interesting dynamics [Fig. 8(b)]. Since the chemical potential $\Delta(H)$ changes at the same rate as $H$, the applied field, the production of ±1 spins favored by $H$ competes equally with the production of 0 spins favored by $\Delta$. For $\alpha > 1$ trajectories, the magnetization $M(H)$ starts to decrease with increasing $H$ for $H > H_{\text{peak}}(R)$, as depicted in Fig. 2 of the main text. In contrast, for $\alpha = 1$, we observe that both the magnetization $M$ and the fraction of zero spins $X$ asymptote to disorder-dependent values $M_{\text{sat}}$ and $X_{\text{sat}}$.

Figure 8. (Color online) Comparison of trajectories with different $\alpha, \Delta_0 > \Delta_c$. The asymptotic ($H \gg H_{\text{peak}}$) dynamics is governed by $\alpha$. $M$ monotonically increases to 1 while $X$ decreases to zero for $\alpha < 1$ trajectories (a). The exact opposite trend is observed for $\alpha > 1$ trajectories (c). For $\alpha = 1$, both $M$ and $X$ increase monotonically and saturate to a value less than 1 (b). The saturation value depends on the disorder.

Figure 9. (Color online) $\alpha = 1$: $X$ (a) and $M$ (b) as functions of the field $H$ for $\alpha = 1$ ($\Delta_0 = 0.9$) trajectories for a few typical disorder strengths, obtained from mean-field calculations. Both order parameters increase monotonically and saturate to a value less than 1. The saturation value depends on $R$. For $M$, the saturation value increases with $R$ while for $X$ it decreases.

Figure 10. (Color online) Numerically obtained asymptotic ($H \gg H_{\text{peak}}$) spin configuration for $\alpha = 1$ trajectories shows microphase separation between spins 1 and 0. $\Delta_0 = 2 > \Delta_c$ and $R = 2$. 

FIG. 8. (Color online) Comparison of trajectories with different \( \alpha, \Delta_0 > \Delta_c \). The asymptotic \( (H \gg H_{\text{peak}}) \) dynamics is governed by \( \alpha \). \( M \) monotonically increases to 1 while \( X \) decreases to zero for \( \alpha < 1 \) trajectories (a). The exact opposite trend is observed for \( \alpha > 1 \) trajectories (c). For \( \alpha = 1 \), both \( M \) and \( X \) increase monotonically and saturate to a value less than 1 (b). The saturation value depends on the disorder.

FIG. 9. (Color online) \( \alpha = 1 \): \( X \) (a) and \( M \) (b) as functions of the field \( H \) for \( \alpha = 1 \) \( (\Delta_0 = 0.9) \) trajectories for a few typical disorder strengths, obtained from mean-field calculations. Both order parameters increase monotonically and saturate to a value less than 1. The saturation value depends on \( R \). For \( M \), the saturation value increases with \( R \) while for \( X \) it decreases.

FIG. 10. (Color online) Numerically obtained asymptotic \( (H \gg H_{\text{peak}}) \) spin configuration for \( \alpha = 1 \) trajectories shows microphase separation between spins 1 and 0. \( \Delta_0 = 2 > \Delta_c \) and \( R = 2 \).
for $H \gg H_{\text{peak}}$. As $R$ increases, $M_{\text{sat}}$ increases while $X_{\text{sat}}$ decreases as shown in Fig. 9.

Simulations of the model [Eq. (4)] in 2D, using zero-temperature Monte Carlo dynamics, show that the asymptotic states for $\alpha = 1$ have a nontrivial spatial distribution of spins. As shown in Fig. 10, there is microphase separation between ±1 and 0 spins. This spatial structure is reminiscent of shear bands observed in shear jamming experiments [3].